$$y_{n+1} = y_n + h\,y'_n + \frac{1}{2}(K_1+K_2)$$

$$y'_{n+1} = y'_n + \frac{1}{2h}(K_1+3K_2) \quad (2.119)$$

The *Runge-Kutta-Nystrom* formula is written as

$$K_1 = \frac{h^2}{2} f(t_n, y_n)$$

$$K_2 = \frac{h^2}{2} f\left(t_n+\frac{2}{5}h,\ y_n+\frac{2}{5}hy'_n+\frac{4}{25}K_1\right)$$

$$K_3 = \frac{h^2}{2} f\left(t_n+\frac{2}{3}h,\ y_n+\frac{2}{3}h\,y'_n+\frac{4}{9}K_1\right)$$

$$K_4 = \frac{h^2}{2} f\left(t_n+\frac{4}{5}h,\ y_n+\frac{4}{5}hy'_n+\frac{8}{25}(K_1+K_2)\right)$$

$$y_{n+1} = y_n+hy'_n+\frac{1}{96}(23K_1+75K_2-27K_3+25K_4)$$

$$y'_{n+1} = y'_n+\frac{1}{96h}(23K_1+125K_2-81K_3+125K_4) \quad (2.120)$$

where the truncation error in y and y' is $0(h^5)$. A formula based on three function evaluations with truncation error $0(h^4)$ is given by

$$K_1 = \frac{h^2}{2} f\left(t_n+\frac{1}{6}h,\ y_n+\frac{1}{6}h\,y'_n\right)$$

$$K_2 = \frac{h^2}{2} f\left(t_n+\frac{1}{2}h,\ y_n+\frac{1}{2}h\,y'_n+\frac{1}{3}K_1\right)$$

$$K_3 = \frac{h^2}{2} f\left(t_n+\frac{5}{6}h,\ y_n+\frac{5}{6}h\,y'_n+\frac{4}{9}K_1+\frac{2}{9}K_2\right)$$

$$y_{n+1} = y_n+h\,y'_n+\frac{1}{16}(10K_1+4K_2+2K_3)$$

$$h\,y'_{n+1} = h\,y'_n+\frac{1}{16}(12K_1+8K_2+12K_3) \quad (2.121)$$

Example 2.5 Solve the initial value problem

$$y'' = (1+t^2)\,y,\ y(0) = 1,\ y'(0) = 0,\ t \in [0, 1]$$

by the Runge-Kutta method (2.119) with $h = 0.1$.

For $n = 0$

$$t_0 = 0,\ y_0 = 1,\ y'_0 = 0$$

$$K_1 = \frac{h^2}{2} f(t_0, y_0) = \frac{(.1)^2}{2}(1+0)\,1 = .005$$

$$K_2 = \frac{h^2}{2} f\left(t_0+\frac{2}{3}h,\ y_0+\frac{2}{3}h\,y'_0+\frac{4}{9}K_1\right)$$

$$= \frac{(.1)^2}{2}\left(1+\frac{4}{9}(.1)^2\right)\left(1+\frac{2}{3}(.1)\,0+\frac{4}{9}(.005)\right)$$
$$= .0050333827$$
$$y_1 = y_0+h\,y_0'+\frac{1}{2}(K_1+K_2)$$
$$= 1+0+\frac{1}{2}(.005+.0050333827)$$
$$= 1.0050167$$
$$y_1' = 0+\frac{1}{2(.1)}(.005+.0151001481) = 0.10050074$$

The exact solution is given by

$$y(t) = e^{t^2/2}$$

The computed solution is listed in Table 2.6.

TABLE 2.6 SOLUTION OF $y'' = (1+t^2)\,y,\ y(0) = 1,\ y'(0) = 0$ BY THE RUNGE-KUTTA METHOD WITH $h = 0.1$

t_n	y_n	y_n'	$y(t_n)$	$y'(t_n)$
0	1	0	1	0
0.1	1.0050167	0.100501	1.0050125	0.100501
0.2	1.0202098	0.204038	1.0202013	0.204040
0.3	1.0460407	0.313802	1.0460279	0.313808
0.4	1.0833046	0.433303	1.0832871	0.433315
0.5	1.1331710	0.566554	1.1331485	0.566574
0.6	1.1972453	0.718298	1.1972174	0.718330
0.7	1.2776552	0.894286	1.2776213	0.894335
0.8	1.3771681	1.101629	1.3771278	1.101702
0.9	1.4993498	1.349266	1.4993030	1.349372
1.0	1.6487762	1.648568	1.6487213	1.648722

2.9.2 Stability analysis

We can discuss the stability and the error analysis of the Runge-Kutta method (2.119) in a manner similar to that adopted in Section 2.5.

Let us consider the differential equation

$$y'' = \alpha y \tag{2.122}$$

subject to the initial conditions

$$y(t_0) = y_0,\ y'(t_0) = y_0',\ t \in [t_0, b]$$

where α is a real number.

We shall discuss the three cases $\alpha = 0, -k^2, k^2$. Using Equation (2.122) into (2.118), we get

$$K_1 = \frac{h^2}{2}\alpha y_n,\ K_2 = \left(\frac{h^2\alpha}{2}+\frac{h^4\alpha^2}{9}\right)y_n+\frac{h^3}{3}\alpha y'_n$$

Substituting the expressions for K_1 and K_2 into (2.119), we find

$$\begin{bmatrix} y_{n+1} \\ y'_{n+1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \end{bmatrix} \tag{2.123}$$

where $$a_{11}=1+\frac{h^2\alpha}{2}+\frac{h^4\alpha^2}{18},\ a_{12}=h+\frac{h^3\alpha}{6}$$

$$a_{21}=\alpha h+\frac{1}{6}h^3\alpha^2,\ a_{22}=1+\frac{h^2\alpha}{2} \tag{2.124}$$

For $\alpha=0$, we have

$$y_{n+1}=y_n+h\,y'_n$$

$$y'_{n+1}=y'_n \tag{2.125}$$

The solution of (2.125) can be written as

$$y'_n=y'_0$$

$$y_n=y_0+nh\,y'_0$$

which is an expected result.

We now consider the case $\alpha=-k^2$; the solutions in this case are oscillating. We, therefore, consider the eigenvalues of the matrix in (2.123), which are given by

$$\lambda_1, \lambda_2=\frac{1}{2}\,[a_{11}+a_{22}\pm[(a_{11}-a_{22})^2+4a_{12}\,a_{21}]^{1/2}] \tag{2.126}$$

Substituting $\alpha=-k^2$ into (2.124) and inserting the resulting values into (2.126), we get

$$\lambda_1, \lambda_2=\frac{1}{2}\left[2-h^2k^2+\frac{h^4k^4}{18}\pm\left[\left(\frac{hk}{18}\right)^2(h^6k^6-36h^4k^4 + 432h^2k^2-1296)\right]^{1/2}\right]$$

$$=\frac{1}{2}\left[2-h^2k^2+\frac{h^4k^4}{18}\pm\left[\left(\frac{hk}{18}\right)^2(h^2k^2-4.44044737)(h^4k^4-2\alpha_1h^2k^2+\alpha_1^2+\gamma^2)\right]^{1/2}\right]$$

where $\alpha_1=15.779763$ and $\gamma=6.5467418$.

Computing λ_1 and λ_2 as functions of h^2k^2, we find that the roots have unit modulus for $0 \leqslant h^2k^2 \leqslant 4.44$. Thus the stability interval of the Runge-Kutta method (2.119) is $0 < h^2k^2 < 4.44$.

where $g(t, y)=py+f(t, y)$ and $p > 0$ is a parameter to be chosen suitably. To solve the differential equation (2.127), we can approximate $g(t, y)$ by a polynomial of an appropriate degree. Here, we will take a quadratic polynomial for $g(t, y)$ with undetermined coefficients. Equation (2.127) may be written as

$$\frac{dy}{dt}=f(t, y)=-p(y-y_n)+A+B(t-t_n)+\frac{C}{2}(t-t_n)^2 \tag{2.128}$$

where (t_n, y_n) is contained in the region of interest. The four constants A, B, C and p can be obtained by determining the value of $f(t, y)$ at four points in the interval $[t_n, t_n+h]$ and solve the resulting equations. We choose the classical Runge-Kutta nodes t_n, $t_n+\frac{h}{2}$, $t_n+\frac{h}{2}$ and t_n+h and put

$$\begin{aligned}
K_1&=hf(t_n, y_n)\\
K_2&=hf\left(t_n+\frac{h}{2}, \bar{y}_{n+1/2}\right), \quad \bar{y}_{n+1/2}=y_n+\frac{1}{2}K_1\\
K_3&=hf\left(t_n+\frac{h}{2}, \bar{\bar{y}}_{n+1/2}\right), \quad \bar{\bar{y}}_{n+1/2}=y_n+\frac{1}{2}K_2\\
K_4&=hf(t_n+h, \bar{y}_{n+1}), \qquad \bar{y}_{n+1}=y_n+K_3
\end{aligned} \tag{2.129}$$

The four equations are

$$\begin{aligned}
K_1&=hA\\
K_2+ph\,\bar{y}_{n+1/2}&=ph\,y_n+Ah+\frac{1}{2}h^2B+\frac{1}{8}h^3C\\
K_3+ph\,\bar{\bar{y}}_{n+1/2}&=ph\,y_n+Ah+\frac{1}{2}h^2B+\frac{1}{8}h^3C\\
K_4+ph\,\bar{y}_{n+1}&=ph\,y_n+Ah+h^2B+\frac{1}{2}h^3C
\end{aligned} \tag{2.130}$$

Solving the equations (2.130), we get

$$\begin{aligned}
hA&=K_1\\
h^2B&=[-3(K_1+ph\,y_n)+2(K_2+ph\,\bar{y}_{n+1/2})\\
&\quad+2(K_3+ph\,\bar{\bar{y}}_{n+1/2})-(K_4+ph\,\bar{y}_{n+1})]\\
h^3C&=4[(K_1+ph\,y_n)-(K_2+ph\,\bar{y}_{n+1/2})\\
&\quad-(K_3+ph\,\bar{\bar{y}}_{n+1/2})+(K_4+ph\,\bar{y}_{n+1})]\\
ph&=-\left[\frac{K_3-K_2}{\bar{\bar{y}}_{n+1/2}-\bar{y}_{n+1/2}}\right]
\end{aligned} \tag{2.131}$$

On integrating (2.128) between the limits t_n to t_{n+1}, we obtain

$$y_{n+1}=y_n+h\,AF_1+h^2\,BF_2+h^3\,CF_3 \tag{2.132}$$

where

$$F_1=\frac{e^{-ph}-1}{-ph},\quad F_2=\frac{e^{-ph}+ph-1}{(ph)^2}$$

$$F_3=\frac{e^{-ph}-\frac{1}{2}(ph)^2+ph-1}{(-ph)^3}$$

$$F_{j+1}=\frac{F_j-\frac{1}{j!}}{(-ph)},\quad j=3,4,\ldots \tag{2.133}$$

Equation (2.132) becomes

$$\begin{aligned}y_{n+1}=y_n+K_1F_1+[&-3(K_1+ph\,y_n)+2(K_2+ph\,\bar{y}_{n+1/2})\\&+2(K_3+ph\,\bar{\bar{y}}_{n+1/2})-(K_4+ph\,\bar{y}_{n+1})]F_2+4[(K_1+ph y_n)\\&-(K_2+ph\,\bar{y}_{n+1/2})-(K_3+ph\,\bar{\bar{y}}_{n+1/2})+(K_4+ph\,\bar{y}_{n+1})]F_3\end{aligned} \tag{2.134}$$

which is the required *Runge-Kutta-Treanor* method. Substituting the values of $\bar{y}_{n+1/2}$, $\bar{\bar{y}}_{n+1/2}$ and $\bar{y}_{n+1}$ from (2.130), the equation (2.134) can be written as

$$\begin{aligned}y_{n+1}=y_n&+\frac{1}{6}(K_1+2K_2+2K_3+K_4)\\&-(ph)^2[(K_2-K_3)F_3+(K_1-4K_2+2K_3\\&+K_4)F_4-4(K_1-K_2-K_3+K_4)F_5)]\end{aligned} \tag{2.135}$$

The value of p is given by

$$\frac{1}{2}ph=-\left(\frac{K_3-K_2}{K_2-K_1}\right) \tag{2.136}$$

The first part in (2.135) is due to the fourth order Runge-Kutta method and the additional term is fifth order and higher in h. It is seen that when the equation (2.134) or (2.135) is used to integrate over the interval where ph is small, the result will be identical with the Runge-Kutta method. If ph is large, a condition where the Runge-Kutta method is known to be unstable then the equation (2.134) gives a far superior solution.

DEFINITION 2.7 An adaptive numerical method is said to be A-stable in the sense of *Dahlquist* if when the method is applied to the equation $y'=\lambda y$, $y(t_0)=y_0$, $\lambda < 0$ with exact initial condition, it gives the true solution which is identical to that of the differential equation for arbitrary h and $p=\lambda$. Here, $f(t, y)=\lambda y$, then $p=-\lambda$ and the equation (2.134) gives

$$y_{n+1}=e^{\lambda h}y_n \tag{2.137}$$

Since $\lambda < 0$ and therefore $y_n\to 0$ when $n\to\infty$ and for any fixed h.

Substituting the above values in (2.144) and using (2.145) to simplify, we have

$$y_{n+1}=y_n+hy'_n+\frac{1}{48}(14K_1+25K_2+9K_3)-\frac{w^2}{2}[(-16K_1+25K_2-9K_3)F_5+(30K_1-75K_2+45K_3)F_6]$$

$$h\,y'_{n+1}=h\,y'_n+\frac{1}{2}(K_1+3K_3)-\frac{w^2}{2}[(-16K_1+15K_2-9K_3)F_4+(30K_1-75K_2+45K_3)F_5] \qquad (2.148)$$

The term in w^2 in (2.148) is the modification to the *Runge-Kutta-Nystrom* method.

DEFINITION 2.9 An adaptive numerical method is said to be *P*-stable if, when the method is applied to the equation $y''=-\lambda y$, $\lambda > 0$, $y(t_0)=y_0$, $y'(t_0)=y'_0$ with exact initial conditions it gives rise to the solution which is identical to that of the differential equation for an arbitrary h and the free parameter is chosen as the square of the frequency.
Here, $p=\lambda$ and the equations (2.144) become

$$y_{n+1}=y_n\cos w+h\,y'_n\frac{\sin w}{w}$$

$$hy'_{n+1}=-y_n w\sin w+h\,y'_n\cos w \qquad (2.149)$$

which may be written in the matrix form as

$$\begin{bmatrix} y_{n+1} \\ hy'_{n+1} \end{bmatrix}=\mathbf{E}(w)\begin{bmatrix} y_n \\ hy'_n \end{bmatrix} \qquad (2.150)$$

where

$$\mathbf{E}(w)=\begin{bmatrix} \cos w & \dfrac{\sin w}{w} \\ -w\sin w & \cos w \end{bmatrix}$$

is a 2×2 matrix. The eigenvalues of the matrix $\mathbf{E}(w)$ are complex and of unit modulii.

Bibliographical Note

There are many text books which deal with the singlestep methods for solving initial value problems of ordinary differential equations. Particularly useful are 33, 46, 93, 113, 161 and 163.

An automatic integration programme based on the Taylor series method for solving initial value problems is given in 94. The Runge-Kutta methods of various order are studied in 23, 25, 174, 175, 209 and 222. The stability of the Runge-Kutta formulas is given in 68. We find the methods with minimum truncation error in 118, 156 and 199, the methods with extended region of stability in 165 and 166, and the error bounds of the methods in 34, 137 and 220.

By an m-fold predifferentiation of the differential equations and a simple transformation of the variables, the Runge-Kutta formulas of high accuracy have been discussed in 82, 83, 145 and 146. The extrapolation algorithms for the initial value problems are established in 21 and 100. The implicit Runge-Kutta methods are given in 22, 24, 28, 208 and 252. The two point Runge-Kutta formulas are found in 27. Using higher order derivatives, the obrechkoff methods are obtained in 177 and 188.

The singlestep methods based upon quadratures and interpolations have been studied in 51, 63, 64, 65, 66, 140 and 223. The Runge-Kutta methods for the system and the higher order initial value problems are discussed in 5, 42, 109, 111, 148, 213, 219 and 260. The adaptive numerical methods are given in 136 and 238.

Problems

1. Obtain the Taylor series solution of the initial value problem

$$y' = 1-2ty, \; y(0) = 0$$

and determine:

(i) t when the error in $y(t)$ obtained from four terms only is to be less than 10^{-6} after rounding.

(ii) The number of terms in the series to find results correct to 10^{-10} for $0 \leqslant t \leqslant 1$.

2. For the solution of the initial value problem

$$y'=p_1(t)\,y+q_1(t),\; y(t_0)=y_0$$

by Taylor's series method, show that

$$y(t+h)=\left(1+h\,p_1+\frac{1}{2}\,h^2 p_2+\dots\right)y(t)+\left(h\,q_1+\frac{1}{2}\,h^2 q_2+\dots\right)$$

where

$$p_{r+1}=p_r'+p_1\,p_r$$

$$q_{r+1}=q_r'+q_1\,p_r$$

3. Apply Taylor's series method of order p to the problem

$$y'=y,\; y(0)=1$$

to show that

$$|\,y_n-y(t_n)\,| \leqslant \frac{h^p}{(p+1)!}\,t_n\,e^{t_n}$$

4. The function $y(t)$ is the solution of the initial value problem

$$y'(t)=f(t,y),\quad t \in [t_0, b]$$
$$y(t_0)=y_0$$

14. Solve the differential equation

$$\frac{dy}{dt}=\frac{t}{y},\ y(0)=1$$

by the Euler method with $h=0.1$ to get $y(0.2)$. Then repeat with $h=0.2$ to get another estimate of $y(0.2)$. Extrapolate these results assuming that errors are proportional to step-size, and compare the derived result to the analytical result.

15. In a computation with Euler's method, the following results are obtained with various step sizes:

$h=2^{-2}$	$h=2^{-3}$	$h=2^{-4}$
2.44141	2.56578	2.63793

Compute a better value by extrapolation.

16. Obtain the Runge-Kutta method of the form

$$K_1=h[1-h\,af_y(y_n)]^{-1}f(y_n)$$
$$y_{n+1}=y_n+W_1K_1$$

for the differential equation $y'=f(y)$, and determine the interval of absolute stability for the equation

$$y'=\lambda y,\ \lambda<0$$

17. Find the Runge-Kutta method of the form

(i)
$$K_1=hf(y_n+a_{11}K_1)$$
$$y_{n+1}=y_n+W_1K_1$$

(ii)
$$K_1=hf(y_n)$$
$$K_2=hf(y_n+a(K_1+K_2))$$
$$y_{n+1}=y_n+W_1K_1+W_2K_2$$

for the initial value problem

$$y'=f(y)$$
$$y(t_0)=y_0$$

and obtain the interval of absolute stability for

$$y'=\lambda y,\ \lambda<0$$

18. Find the order of the implicit Runge-Kutta method

$$y_{n+1}=y_n+\frac{1}{6}h[4f(t_n,y_n)+2f(t_{n+1},y_{n+1})+hf'(t_n,y_n)]$$

and determine its interval of absolute stability.

19. The use of the fourth order implicit Runge-Kutta method (2.66) requires the solution of the nonlinear equations at each step. They can be solved by an iteration method. Find the condition for the convergence of the iteration method.